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https://doi.org/10.1007/s11750-020-00572-z

ORIGINAL PAPER

The ordered median tree of hubs location problem

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Received: 3 February 2020 / Accepted: 26 May 2020

Abstract
In this paper, we propose the Ordered Median Tree of Hub Location Problem (OMTHL). The OMTHL is a single-allocation hub location problem where p hubs must be placed on a network and connected by a non-directed tree. Each non-hub node is assigned to a single hub and all the flow between origin–destination pairs must cross at least one hub. The objective is to minimize the sum of the ordered weighted averaged collection and distribution costs plus the sum of the interhub flow costs. We present different MILP formulations for the OMTHL based on the properties of the Minimum Spanning Tree Problem, the ordered median optimization and on the different ways of modeling flow within the network. Given that ordered median hub location problems are rather difficult to solve, we have improved the OMTHL solution performance by introducing covering variables in two valid reformulations. In addition, we have developed two pre-processing phases to reduce the size of these formulations. We establish an empirical comparison between these new formulations and we also provide enhancements that together with a proper formulation allow to solve medium-size instances on general random graphs.

Keywords Minimum spanning tree · Hub-and-spoke models · Ordered median

Mathematics Subject Classification 90B80 · 90C11

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Published online: 22 June 2020

1 Introduction

Hub-and-spoke problems have a great importance in transportation and telecommunication systems in which several origin–destination points exchange flows. In such models, the key feature is to connect each pair via specific subsets of links to consolidate and distribute the flows to reduce costs based on the economy of scale of intermediate connections. Therefore, Hub Location Problems (HLPs) integrate two decision levels: location of facilities (hubs) to consolidate deliveries and network design to determine the routes for the flow between different origin–destination pairs to optimize some performance measure. Hub location has become an important area of research in the field of Supply Chain Networks and one can find in the last years a large number of specialized publications in the field (Boland et al. 2004; Cánovas et al. 2007; Campbell 1996; Campbell et al. 2007; Contreras et al. 2009; Ernst and Krishnamoorthy 1999; Hamacher et al. 2004; Labbé and Yaman 2004, 2008; Labbé et al. 2005; Marín 2005a, b; Marín et al. 2006; Rodríguez-Martín and Salazar-González 2008; Yaman 2005; Bollapragada et al. 2006; Kara and Tansel 2000, 2003; Kratica and Stanimirović 2006; Meyer et al. 2009; Tan and Kara 2007; Wagner 2008; Taherkhani and Alumur 2019; Blanco and Marín 2019; Fernández and Sgalambro 2020); as well as several interesting surveys (Campbell et al. 2002; Alumur and Kara 2008; Campbell and O'Kelly 2012; Farahani et al. 2013; Contreras and O'Kelly 2019).

Traditionally, many hub location models assume that the hubs are fully interconnected, that is to say, there exists a link connecting any pair of hubs, which can be used by applying the corresponding discount factor. However, it is widely accepted that there exist many applications in which the backbone network (i.e., the network connecting the facilities) is not fully interconnected (see O'Kelly and Miller 1994). Several HLPs considering incomplete hub level networks have thus been studied. These problems can be seen from a hub arc location perspective (see Campbell et al. 2005a, b; Contreras and Fernández 2014), in which the location of a set of hub arcs and their associated hub nodes is considered. Motivated by specific applications, some of these models require the hub-level network to have a particular topological structure, such as cycles (Contreras et al. 2017; Lee et al. 1993), stars (Labbé and Yaman 2008), trees (Contreras et al. 2009, 2010, Martins de Sá et al. 2013), or lines (Martins de Sá et al. 2015). Some other models do not even require the hub arcs to define a single connected component (Campbell et al. 2005a; Contreras and Fernández 2014).

As mentioned, an example of a hub location model that does not require full interconnection between hubs is the Tree of Hub Location problem (THL), introduced by Contreras et al. (2009, 2010). Similarly, in this paper, we use a tree as backbone network connecting the hubs in a different problem. It is a single-allocation hub location problem where a fixed number of hubs have to be located on a network, with the particularity that it is required that the hubs are connected by means of a (non-directed) tree. The THL is defined on a graph where it is assumed that for each pair of nodes, there exists a known amount of flow that must be sent through the network.

Recently, another feature, namely weighted averaging objective functions, has also been incorporated to the analysis of HLPs (Puerto et al. 2011; Ramos 2012; Puerto et al. 2013, 2016). It has been recognized as a powerful tool from a modeling point of view because it allows to distinguish the roles played by the different entities participating in a hub-and-spoke network inducing new type of distribution patterns, see Rodríguez-Martín and Salazar-González (2008), Fonseca et al. (2010), Kalcsics et al. (2010a) and Kalcsics et al. (2010b). Each one of the components of any origin–destination delivery path gives rise to a cost that is weighted by different compensation factors depending on the role of the entity that supports the cost. This adds a "sorting" problem to the underlying hub location problem. The objective is to minimize the total transportation cost of the flows between each origin–destination pair after applying rank-dependent compensation factors on the transportation costs.

In this paper, we consider the Ordered Median Tree of Hub Location Problem (OMTHL) that is a single-allocation hub location problem, where p hubs must be placed on a network and connected by a non-directed tree. Each node must be allocated to one single hub and all the flow from/to this node must leave/enter it through its allocated hub. Excepting the arcs that connect each node with its allocated hub, the only arcs that can be used for routing the flows must be links connecting hubs. There is a unit transportation cost associated with each arc. As usual, if hubs are located at both end-nodes of the arc, a discount factor is applied to the unit cost. The objective is to minimize the operation costs of the system (collection and distribution flow costs), which depend on the amount of flow that circulates through the arcs connecting origin/destination to hubs using the ordered median objective function, and on the links between hubs to which the discount factor is applied (inter-hub flow costs).

As mentioned above, our hub location model has two main modeling aspects: (1) the tree structure defined by the hub network and (2) the rank-dependent compensation factors applied to the operation costs of the system (collection and distribution flow costs) through the ordered median function. Several potential applications justify the analysis of the proposed model. As for the first modeling aspect, it is worth mentioning that models where facilities are connected by means of a tree arise when the cost of the links between facilities is very high, and as a consequence full interconnection is prohibitive. Specific applications of such problems arise mostly in telecommunications (see Hu 1974; Nguyen and Knippel 2007) and in transportation (see, for instance, the recent work of Chen et al. (2008) for an excellent description of the practical relevance of tree-backbone problems in small package delivery). On the other hand, regarding the rank-dependent weights applied to the operation costs, its application naturally arise when these weights can be seen as compensation factors that try to diminish unfair situations of the origin/destination sites with respect to the distribution system (tree of hubs). The reader may note that we are simultaneously making decisions on placing hubs that define the intermediate distribution system, and establishing the delivery paths from origin sites to final destination. Thus, a solution that is good for the system (the entire supply chain) might not be acceptable for single parties if in that solution their costs to reach the system are too high relative to similar costs for other parties. In this situation some compensation to unhappy sites may be expected to prevent those sites from not using the system. The goal of our rank-dependent weights is to compensate unfair situations, such as those described above. One specific application of the THLP is the design of a backbone underground train network in a city. Since the underground is a public service, the goal is that each district is close to at least one metro station; therefore, in our objective function, long distances will be penalized with the largest weights, for instance using k-centrum criteria.

The remainder of the paper is organized as follows. In Sect. 2, we formally define the OMTHL and a scheme that relates this problem with others well known in the literature. Section 3 presents the catalog of formulations that we study for the OMTHL. Section 4 develops some preprocessing phases for fixing variables and some valid inequalities are developed to enhance the initial formulations. The empirical performance of the resulting OMTHL formulations is analyzed in Sect. 5, where we present extensive numerical results and a comparison of this formulations for different particular cases. Finally, some conclusions are summarized in Sect. 6.

2 Problem description and subproblems

2.1 Notation and definition

In this section, we formally introduce the OMTHL and fix the notation for the rest of the paper. The OMTHL is an extension of several classical hub location problems combining that the interhub network must be a spanning tree (see, e.g., Contreras et al. 2009) and *distribution* plus *collection* costs (the costs from the origin to the hub system plus from the hub system to the origin) are aggregated using an ordered median objective function (see e.g. Puerto et al. 2011). We proceed next with the notation. Let V be the set of demand nodes which exchange flow that are also assumed to be candidates to become hubs. For all origin–destination pairs ij, with $i, j \in V$, there exists an amount of flow w_{ij} to be delivered from i to j through at least one hub. Let also $O_i = \sum_{j \in V} w_{ij}$ and $D_i = \sum_{j \in V} w_{ji}$ be the total flow originating at and entering in node $i \in V$, respectively. The model assumes that single allocation, i.e., each node has a common hub for its outgoing flow and incoming flow; so once it is fixed, every path connecting each origin–destination pair ij is unique. When there is only one hub present in such a path, there are no inter-hub links; otherwise, the inter-hub connection must use the edges of the installed tree of hubs. The use of each delivery path induces two types of per unit flow costs: (1) the inter-hubs delivery cost and (2) the distribution and collection costs. To the former cost, a discount factor $0 \leq \alpha \leq 1$ is applied representing the economy of scale. Thus, each unit of flow that is sent from a non-hub node i to a hub k (likewise from a hub m to a non-hub j) induces a cost $c_{ik} \geq 0$ ($c_{mj} \geq 0$); whereas when it goes between two hub nodes k and m, the cost induced is αc_{km}. No further hypothesis on symmetry of costs or triangular inequality is assumed.

In addition, this model compensates distribution and collection costs using the scaling factor parameters $\lambda = (\lambda_1, \dots, \lambda_{|V|})$ (Puerto et al. 2011, 2013, 2016 and see also Boland et al. 2006; Kalcsics et al. 2010a; Marín et al. 2009, 2010; Nickel and

Table 1 Notation introduced for the OMTHL problem

V	Set of nodes and hub candidates
i, j, k, m	Indexes for the network nodes
w_{ij}	Amount of flow to be delivered from i to j through at least one hub
O_i	Total flow originated at node $i \in V$
D_i	Total flow entering in node $i \in V$
α	Discount factor applied to the inter-hubs delivery representing the economy of scale
c_{ik}	Cost induced by each unit of flow that is sent from a non-hub node i to a hub k
αc_{km}	Cost induced by each unit of flow that is sent between two hub nodes k and m
$\ell \in V$	Index for the ℓ-th position of the sorted sequence of distribution and collection costs
λ_ℓ	Scaling factor for the ℓ-th distribution and collection cost
p	Fixed number of hubs to locate

Puerto 2005 for different ordered median location models). Indeed, if in a solution, a node i sends and receives commodity from a hub k and this cost, namely $(c_{ik}O_i + c_{ki}D_i)$, was ranked in the ℓ-th position among these type of costs, then this term would be scaled by λ_ℓ, i.e., the corresponding objective function component would be $\lambda_\ell(c_{ik}O_i + c_{ki}D_i)$. For the sake of understandability, we summarize the introduced notation in Table 1.

Observe that depending on the choices of the λ-vector, we can obtain different criteria to account for the costs in the objective function. For instance, if $\lambda = (0, \dots, 0, 1, \overset{k}{\dots}, 1)$ is considered, the first component of the objective function would be the sum of the k-largest costs (k-centrum). This usually provides different solutions or different allocation patterns for problems with different λ, even though the optimal solution gets the same set of open hubs (see Puerto et al. 2011 for further details).

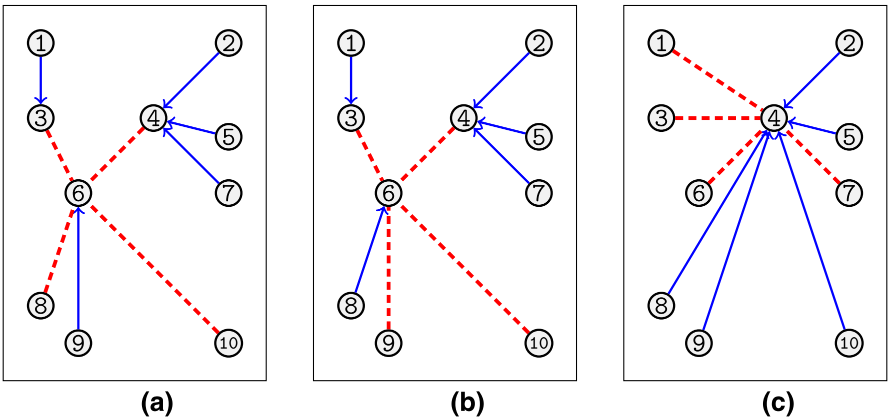

Fig. 1 Three different OMTHL solutions according to median (**a**), k-centrum (**b**) and k-trimmed mean criterion (**c**), considering $\alpha = 0.8$, Euclidean distances and equal origin–destination flows

With the above elements, the OMTHL consists of locating p hubs and setting an interhub network linking them with a non-directed tree. Moreover, each non-hub node must be allocated to one single hub in such a way that the rank-dependent compensated costs of the non-inter-hub deliveries (collection and distribution costs) plus the discounted cost of the inter-hubs deliveries are minimized. Figure 1 depicts different OMTHL solutions according to (a) the median $\lambda = (1, \overset{10}{\ldots}, 1)$, (b) k-centrum $\lambda = (0, \overset{6}{\ldots}, 0, 1, \overset{4}{\ldots}, 1)$ and (c) k-trimmed mean criterion $\lambda = (0, 0, 0, 1, 1, 1, 1, 0, 0, 0)$, considering $\alpha = 0.8$, Euclidean distances and equal origin–destination flows.

Our first observation is that the OMTHL is an $\mathcal{NP}$-hard problem which comes from the hardness results of the different combinatorial problems that give rise to the OMTHL. By Contreras et al. (2010) its proof is straighforward since THL is a particular case of OMTHL.

In the following, we will present several integer programming formulations for the OMTHL. Depending on the cases, we use different sets of variables, mostly borrowed from some previous formulations on Hub Location Problems.

2.2 OMTHL subproblems

The OMTHL is a complex network design problem that involves a number of components each of which is by itself a hard combinatorial optimization problem. The OMTHL has as subproblems different well-known problems of the literature. Starting with the classical p-median location problem (PMED) three different elements can be added to model new features.

1. Connectivity: A connected structure can be imposed among the chosen facilities
2. Flow: Optimizing the location of hub facilities and spokes allocations that minimizes the sum of distribution and collection costs plus the weighted costs of inter-hub connections using a connected structure, subject to a budget constraint for the inter-hub connected structure.
3. Sorting: Transportation costs between nodes and facilities are sorted in non decreasing order and each term in this sequence is weighted with a correction factor.

In this way, the connectivity imposed to PMED gives rise to the p-median location problem with inner Connected structure (PMEDC). When the connected structure is a tree, we define the p-median location problem with inner Tree structure (PMEDT) that is closely related to the problem known in the literature as the connected facility location problem (Gollowitzer and Ljubić 2011). In addition, flow can be included into PMEDC and/or PMEDT giving rise to the network Hub Location problem (HL) and the Tree of Hubs Location problem (THL) (Contreras et al. 2009), respectively.

Besides, sorting can be included in PMED giving rise to the well-known Ordered Median location problem (OM) (Nickel and Puerto 2005). In this way, adding also connectivity onto OM, gives rise to the Ordered Median location problem with inner Connected structure (OMC). When the connected structure is a tree, we define the

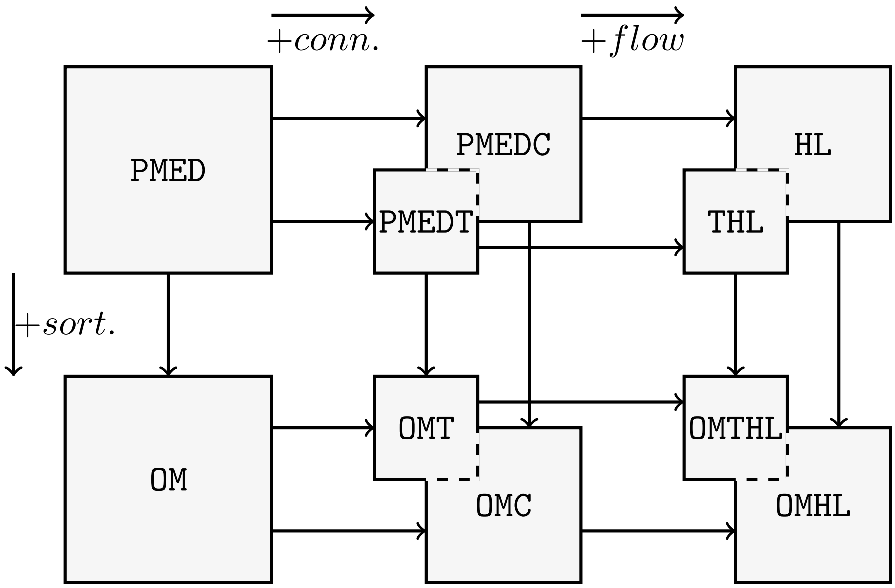

Fig. 2 Diagram of relationships among the different models

Table 2 Sets of OMTHL constraints that are activated for each OMTHL subproblem

OMTHL	PMED	PMEDC	PMEDT	HL	THL	OM	OMC	OMT	OMHL
(1a)	$\alpha = 0, \lambda = \mathbf{1}$	$\alpha = 0, \lambda = \mathbf{1}$	$\alpha = 0, \lambda = \mathbf{1}$	$\lambda = \mathbf{1}$	$\lambda = \mathbf{1}$	$\alpha = 0$	$\alpha = 0$	$\alpha = 0$	
(1b)		✓	✓		✓			✓	
(1c)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(1d)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(1e)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(1f)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(1g)				✓	✓				✓
(1h)				✓	✓				✓
(1i)						✓	✓	✓	✓
(1j)						✓	✓	✓	✓
(1k)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(1l)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(1m)	✓	✓	✓	✓	✓	✓	✓	✓	✓
(1n)		✓	✓	✓	✓		✓	✓	✓
(1o)				✓	✓				✓

Ordered Median location problem with inner Tree structure (OMT). In addition, flow can be included into OMC and/or OMT giving rise to the Ordered Median Hub Location problem (OMHL) (Puerto et al. 2011) and the Ordered Median Tree of

Hubs Location problem (OMTHL), respectively. The reader is referred to Fig. 2 to have an overview of the different considered problems and their relationships.

OMTHL subproblems can be described explicitly according to a general OMTHL formulation, namely F^{4i}, that is presented in Sect. 3.1. Table 2 displays the set of constraints that are activated from the OMTHL formulation for each OMTHL subproblem.

Concerning the sizes that one could expect to solve for the OMTHL, we note in passing that the OMHL is the most similar problem to the OMTHL. Since OMHL has been solved to optimality for non-complete graphs (AP data) up to 30 nodes (see Puerto et al. 2011, 2013), that should be the size of the largest set of instances that could be expected to be solved for the OMTHL. As we shall show later, this is confirmed by our computational experiments in Sect. 5.

3 OMTHL formulations

3.1 A 4-index MILP formulation for the OMTHL

The first formulation that we present is based on the use of four-index variables to trace the paths followed by the flow between origins and destinations. This formulation is based on the ones presented by Contreras et al. (2009) for the Tree of Hubs Location Problem without order. The main characteristic of this formulation was that it provides very good LP-bounds. To present this formulation, we first define the following families of variables for any $i, j, k, m \in V : i < j$,

$$x_{ikmj} = \begin{cases} 1, & \text{if the flow that goes from origin } i \text{ to destination } j,\ i < j, \text{ passes through the arc} \\ & (k, m) \text{ connecting hubs } k \text{ and } m, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that if this variable takes value 1, it also means implicitly that the flow that goes from j to i, $i < j$ passes through the arc (m, k).

Next, to model the tree of hubs, we use a standard flow formulation as for instance the one given in Contreras et al. (2010). That type of formulation needs binary variables indicating whether or not an arc is used in the tree. For any $k, m \in V : k < m$, let

$$y_{km} = \begin{cases} 1, & \text{if the edge } \{k, m\} \text{ connects hubs } k \text{ and } m > k, \\ 0, & \text{otherwise.} \end{cases}$$

We represent the allocation of node i to hub k simultaneously with the sorted position of the associated distribution and collection cost using the following binary variables for any $i, k, \ell \in V$:

$$z_{ik}^{\ell} = \begin{cases} 1, & \text{if a node } i \text{ is allocated to a hub } k, \text{ and the cost } c_{ik}O_i + c_{ki}D_i \text{ is the } \ell\text{-th smallest,} \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we define the aggregated version of z_{ik}^{ℓ} variables, as $z_{ik} := \sum_{\ell \in V} z_{ik}^{\ell}$, for modeling purposes (see Table 2):

$$z_{ik} = \begin{cases} 1, & \text{if a node } i \text{ is allocated to a hub } k, \\ 0, & \text{otherwise.} \end{cases}$$

Parameters $\lambda = (\lambda_1, \dots, \lambda_{|V|})$ compensate distribution and collection costs by assigning the scaling factor λ_ℓ to each cost $(c_{ik}O_i + c_{ki}D_i)$ ranked in the ℓth position. The reader may observe that some of the previous variables were already used in Puerto et al. (2016) to give an alternative model for the OMH.

The four-index formulation for OMTHL is:

$$F^{4i}: \min \sum_{\ell \in V} \sum_{i \in V} \sum_{k \in V} \lambda_\ell (c_{ik}O_i + c_{ki}D_i) z_{ik}^{\ell} + \alpha \sum_{k \in V} \sum_{m \in V : m \neq k} \sum_{i \in V} \sum_{j \in V : i < j} (c_{km}w_{ij} + c_{mk}w_{ji}) x_{ikmj} \tag{1a}$$

$$\text{s. t.} \sum_{k \in V} \sum_{m \in V : m > k} y_{km} = p - 1 \tag{1b}$$

$$\sum_{k \in V} z_{ik} = 1 \quad i \in V \tag{1c}$$

$$\sum_{k \in V} z_{kk} = p \tag{1d}$$

$$z_{km} + y_{km} \leq z_{mm} \quad k, m \in V : k < m \tag{1e}$$

$$z_{mk} + y_{km} \leq z_{kk} \quad k, m \in V : k < m \tag{1f}$$

$$x_{ikmj} + x_{imkj} \leq y_{km} \quad i, j, k, m \in V : k < m \wedge i < j \tag{1g}$$

$$z_{jk} + \sum_{m \in V : m \neq k} x_{ikmj} = z_{ik} + \sum_{m \in V : m \neq k} x_{imkj} \quad i, j, k \in V : i \neq k \wedge i < j \tag{1h}$$

$$\sum_{i \in V} \sum_{k \in V} z_{ik}^{\ell} \leq 1 \quad \ell \in V \tag{1i}$$

$$\sum_{i \in V} \sum_{k \in V} z_{ik}^{\ell} (c_{ik}O_i + c_{ki}D_i) \leq \sum_{i \in V} \sum_{k \in V} z_{ik}^{\ell+1} (c_{ik}O_i + c_{ki}D_i) \quad \ell \in V : \ell < |V| \tag{1j}$$

$$z_{ik} = \sum_{\ell \in V} z_{ik}^{\ell} \quad i, k \in V \tag{1k}$$

$$z_{ik}^{\ell} \in \{0, 1\} \quad \ell, i, k \in V \tag{1l}$$

$$z_{ik} \in \{0, 1\} \quad i, k \in V \tag{1m}$$

$$y_{km} \in \{0, 1\} \quad k, m \in V : k < m. \tag{1n}$$

$$x_{ikmj} \geq 0 \quad i, j, k, m \in V : k \neq m \wedge i < j. \tag{1o}$$

Objective function (1a) ensures that the total cost is minimized. The first component accounts for the compensated sorted distribution and collection costs; while, the second one is the interhub transportation cost. Constraints (1b) ensure that the network structure that is induced by the y variables contains $p-1$ edges. Constraints (1c) ensure that each origin is assigned exactly to one hub. Constraint (1d) fixes the number p of hubs to be installed. Constraints (1e) and (1f) avoid an edge to be used simultaneously as an allocation and as an interhub connection. Moreover, they also ensure that the access nodes to the tree of hubs and the nodes defining this tree should be open hubs. Constraints (1g) ensure that flow can only be routed through installed links. Constraints (1h) guarantee the conservation of the flow between each origin–destination pair. For a given pair jk, the left-hand side of the equality adds up the flow incoming node k directly from node i (if i is not a hub and it is allocated to k), plus the flow with origin at i incoming node k from another hub m. The right-hand side of the equality adds up the flow with origin in i going from hub k to another hub (if k is a hub), plus the flow with origin in i and destination in any non-hub node allocated to hub i (again if k is a hub). These constraints together with (1b) ensure that the network structure that is induced by the y variables is a tree. Constraints (1i) prevent that a sorted position is assumed by more than one allocation cost. Constraints (1j) ensure that the correct order of costs in the first term of the objective function is assured. Finally constraints (1k) relate z_{ik} and z_{ik}^{ℓ}. When solving the OMTHL, z_{ik} variables are replaced by the aggregation of z_{ik}^{ℓ}, giving rise to a formulation with a smaller number of variables.

3.2 A 3-index MILP formulation for the OMTHL

This subsection presents another integer programming formulation for the OMTHL using now a set of 3-index continuous variables to represent routes between origins and destinations (see Puerto et al. 2011, 2013, 2016). These variables are defined as the amount of flow that is sent from origin $i \in V$ that traverses arc (k, m) for any $k, m \in V$:

$x_{ikm} =$ amount of flow with origin at i that traverses the hub arc (k, m).

We also use the previously defined y- and z-variables. Hence, the formulation is given by:

$$F^{3i} : \min \sum_{\ell \in V} \sum_{i \in V} \sum_{k \in V} \lambda_{\ell} (c_{ik} O_i + c_{ki} D_i) z_{ik}^{\ell} + \alpha \sum_{k \in V} \sum_{m \in V : m \neq k} \sum_{i \in V} c_{km} x_{ikm} \tag{2a}$$

$$\text{s. t.}(1b) - (1f)$$

$$x_{ikm} + x_{imk} \leq O_i y_{km} \quad i,k,m \in V : k < m \tag{2b}$$

$$O_i z_{ik} + \sum_{m \in V: m \neq k} x_{imk} = \sum_{j \in V} w_{ij} z_{jk} + \sum_{m \in V: m \neq k} x_{ikm} \quad i,k \in V : i \neq k \tag{2c}$$

$$(1i) - (1n)$$

$$x_{ikm} \geq 0 \quad i,k,m \in V : k \neq m. \tag{2d}$$

As it is usual in formulations based on aggregated flows the LP bound tends to be weak. To enhance this bound, one can strengthen constraints (2b) using a better upper bound on the right-hand side. It is clear that the flow sent from node i that will traverse edge (k, m) will be upper bounded by O_i minus the flow that visits only the node i (w_{ii}), minus the flow that visits the first node of that edge, namely $\min\{w_{ik}, w_{im}\}$. Thus, O_i can be replaced in constraints (2b) by

$$\begin{aligned}
x_{ikm} + x_{imk} &\leq (O_i - w_{ii} - \min\{w_{ik}, w_{im}\}) y_{km}, \quad \forall i,k,m \in V : k < m, i \neq k, m, \\
x_{iim} &\leq (O_i - w_{ii}) y_{im}, \quad \forall i,m \in V : m > i, \\
x_{iim} &\leq (O_i - w_{ii}) y_{mi}, \quad \forall i,m \in V : m < i.
\end{aligned}$$

There are also some easy variable fixing for variables as for instance

$$x_{iki} = 0,$$

since i being an open hub does not send flow to itself through any intermediate edge (k, i). All this standard preprocessing has been considered as part of the original formulation of OMTHL in our computational tests.

Some valid inequalities for this model are based on the extension of the mixed-cut inequality (see Ortega and Wolsey 2003) adapted to the THL by Contreras et al. (2010).

- For $m \in V$ and $p > 1$,

$$\sum_{k<m} y_{km} + \sum_{m<k} y_{mk} \geq z_{mm}, \tag{3}$$

  is a valid inequality. Indeed, it represents that a hub cannot be isolated, with respect to the other hubs, whenever the number of hubs to be located is greater than 1, i.e., at least a hub arc is incident in any hub.
- For $i, m \in V$ and $F \subset V \backslash \{m\}$,

$$\sum_{k \in V \backslash F} x_{ikm} + w_{im} \left(\sum_{\substack{k \in F \\ k<m}} y_{km} + \sum_{\substack{k \in F \\ k>m}} y_{mk} \right) \geq w_{im}(z_{mm} - z_{im}). \tag{4}$$

This valid inequality means that if a hub is located at site m and node i is not assigned to hub m, then the flow from origin i to destination m, w_{im}, is sent by a hub arc not included in F or crossing from F to $V \backslash F$.

- For $i, m \in V$ and $F \subset V \backslash \{m\}$, $J \subset V \backslash \{i, m\}$,

$$\sum_{k \in V \backslash (F \cup \{m\})} x_{ikm} + \left(\sum_{j \in J \cup \{m\}} w_{ij} \right) \left(\sum_{\substack{k \in F \\ k<m}} y_{km} + \sum_{\substack{k \in F \\ k>m}} y_{mk} \right) \geq \sum_{j \in J \cup \{m\}} w_{ij}(z_{jm} - z_{im}). \tag{5}$$

This valid inequality means that the sum of flows with origin in i and destination j with $j \in J \cup \{m\}$ coming via hub m, where m is not the first hub visited by this flow is smaller than or equal to the sum of the flow with origin i and destination j with $j \in J \cup \{m\}$, crossing from F to $V \backslash F$ plus the flow with origin i crossing a hub arc out of F.

Clearly, inequalities (4) are dominated by (5). Therefore, the former will not be considered in our computational experiments. In addition, we observe that (5) are an adaptation of the family of inequalities (Contreras et al. 2010, Proposition 4) and therefore, they can be separated by an easy adaptation of the procedure described in that paper, that we have used in our numerical tests.

3.3 A covering MILP formulation for the OMTHL

It has been previously observed (see Puerto et al. 2011) that ordered median hub location problems with the variables used in the previous section are rather difficult to solve and one can improve performance by introducing covering variables. The goal of this section is to find valid reformulations of the OMTHL using this alternative rationale.

Sorting the different delivery costs values $c_{ik}O_i + c_{ki}D_i$ for $i, k \in V$, in increasing order, we get the ordered cost sequence:

$$c_{(1)} := 0 < c_{(2)} < \cdots < c_{(|H|)} := \max_{1 \leq i,k \leq |V|} \{c_{ik}O_i + c_{ki}D_i\},$$

where $|H|$ is the number of different non-null elements of the above cost sequence and $H := \{1, \dots, |H|\}$.

The first covering formulation extends (1a)–(1o) and it is based on variables y_{km}, x_{ikmj}, z_{ik} and a new set of variables for any $\ell \in V$, $h \in H$:

$$u_{\ell h} = \begin{cases} 1, & \text{if the } \ell \text{ th assignment cost is at least } c_{(h)}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the covering formulation using four-index variables is:

$$F^{4ic} : \min \sum_{\ell \in V} \sum_{h \in H} \lambda_\ell u_{\ell h}(c_{(h)} - c_{(h-1)}) + \alpha \sum_{k \in V} \sum_{m \in V: m \neq k} \sum_{i \in V} \sum_{j \in V: i<j} (c_{km}w_{ij} + c_{mk}w_{ji})x_{ikmj} \tag{6a}$$

$$\text{s. t.}(1b) - (1h)$$

$$\sum_{\ell\in V} u_{\ell h} = \sum_{i\in V}\ \sum_{k\in V: c_{ik}O_i + c_{ki}D_i \geq c_{(h)}} z_{ik} \quad h \in H \tag{6b}$$

$$u_{\ell h} \leq u_{\ell+1,h} \quad h\in H; \ell\in V : \ell < |V| \tag{6c}$$

$$u_{\ell h} \in \{0,1\} \quad h \in H; \ell \in V \tag{6d}$$

$$(1m) - (1o)$$

The corresponding covering formulation using three-index variables, F^{3ic}, can be defined analogously and it is left to the reader.

4 Improvements

4.1 Variable fixing

Given that the most promising formulations correspond to the covering models, in this section, we develop some preprocessings to reduce the size of these formulations in terms of the defined variables. These variable fixing procedures are based on non straightforward adaptations of some arguments already used in Puerto et al. (2011) and Puerto et al. (2013) for a different formulation. The intuitive idea behind these preprocessing phases is to fix to 1 u-variables in the left-bottom part of the u-matrix (i.e., for small values of h, hopefully the number of assignments at cost smaller than $c_{(h)}$ should be small) and to 0 in the right-top part of the u-matrix (i.e., for large values of h, hopefully, the number of assignments at cost smaller than $c_{(h)}$ should be high).

4.1.1 Preprocessing for fixing variables to 1

To fix u_{ih}-variables to 1 for a given $h \in H$, we deal with an auxiliary problem that maximizes the number of origin-first hub allocations satisfying $(c_{ik}O_i + c_{ki}D_i)c_{ik} \leq c_{(h-1)}$ which is equivalent to the maximum number of variables u_{ih} that can assume value of 1.

Using the variables previously defined, the formulation of this problem is:

$$F^{f1} : \max \bar{h}^1_h := \sum_{i\in V}\sum_{k\in V} z_{ik} \tag{7a}$$

$$\text{s. t.} \sum_{k\in V} z_{ik} \leq 1 \quad i \in V \tag{7b}$$

$$\sum_{k\in V} z_{kk} = p \tag{7c}$$

$$z_{ik} \leq z_{kk} \quad i,k \in V : i \neq k \tag{7d}$$

$$(c_{ik}O_i + c_{ki}D_i)z_{ik} \leq c_{(h-1)} \quad i,k \in V \tag{7e}$$

$$z_{ik} \in \{0,1\} \quad i,k \in V. \tag{7f}$$

If $\bar{h}^1_h$ is the optimal value of problem above, since there are |V| origin-first hub allocations, the number of allocations satisfying $(c_{ik}O_i + c_{ki}D_i)c_{ik} \leq c_{(h-1)}$ must be necessarily greater than or equal to $|V| - \bar{h}^1_h + 1$, or equivalently, in any feasible solution of a Covering formulation

$$u_{\ell h} = 1, \quad \ell \in V : \ell > \bar{h}^1_h.$$

4.1.2 Preprocessing for fixing variables to 0

Under a similar rationale to the one used in the previous section, we try to fix as many u_{ih}-variables to 0 as possible, for a given $h \in H$. In this case, we deal with an auxiliary problem that maximizes the number of origin-first hub allocations satisfying $(c_{ik}O_i + c_{ki}D_i)c_{jk} \geq c_{(h)}$. In conclusion, this auxiliary problem provides the minimum number of zeros that the hth column of the u-matrix must have. Using the variables defined previously, the formulation of this problem is:

$$F^{f0} : \max H2_h := \sum_{i\in V}\sum_{k\in V} z_{ik} \tag{8a}$$

$$\text{s. t.} \sum_{k\in V} z_{ik} \leq 1, \quad i \in V \tag{8b}$$

$$\sum_{k\in V} z_{kk} = p \tag{8c}$$

$$z_{ik} \leq z_{kk} \quad i,k \in V : i \neq k \tag{8d}$$

$$(c_{ik}O_i + c_{ki}D_i) \geq c_{(h)}z_{ik} \quad i,k \in V \tag{8e}$$

$$z_{ik} \in \{0,1\} \quad i,k \in V \tag{8f}$$

Therefore, if $H2_h$ is the optimal value of problem above, the h-th column of the u-matrix must have at least $|V| - H2_h$ zeros, i.e., in any feasible solution of the Covering formulation:

$$u_{\ell h} = 0, \quad \forall \ell = 1, \dots, |V| - H2_h.$$

As it is shown in the next section, the use of the previous valid inequalities plus these variable fixing procedures help significantly in solving the OMTHL.

5 Computational results

This section reports on the results of some computational experiments we have run, to empirically compare the proposed formulations. We have studied the OMTHL combining the different formulations proposed, namely F^{4i} (see Sect. 3.1), F^{3i} (see Sect. 3.2), F^{4ic} and F^{3ic} (see both in Sect. 3.3). Given that the sizes of our formulations are a critical issue, we provide in Table 3 theoretical sizes values in terms of constraints (#*c*), variables (#*v*), and binary variables (#*b*). In addition, we also provide in Table 4 particular figures of these sizes for complete graphs when $|V| \in \{20, 30, 40, 50\}$ and $|H| = 200$.

For our computational experiments, we have considered complete networks of sizes (number of nodes) $|V| \in \{20, 30, 40, 50\}$ and integer costs assigned to edges randomly generated following a uniform distribution in [1, 100]. We have chosen unitary flows sent among each origin–destination pair. Unitary flows imply constant values for $O_i = D_i = |V| - 1$ and also imply $|H| \leq 200$ since $c_{ik}, c_{ki} \in \{1, \dots, 100\}$ (there are at most 200 different collection and distribution costs $c_{ik}O_i + c_{ki}D_i$ to be sorted). In addition, the discount factor α and the number of hubs parameter p vary according to $\alpha \in \{0.5, 0.8\}$, $p \in \{3, 5, 8\}$, respectively. For this set of instances, we have tested three different criteria for the λ values:

- median criterion: $\lambda = (\underbrace{1, 1, \dots, 1}_{|V|})$,
- k-centrum criterion: $\lambda = (\underbrace{0, \dots, 0}_{\lfloor \frac{2}{3}|V| \rfloor}, \underbrace{1, \dots, 1}_{|V| - \lfloor \frac{2}{3}|V| \rfloor})$,
- k-trimmed mean criterion: $\lambda = (\underbrace{0, \dots, 0}_{\lfloor \frac{1}{3}|V| \rfloor}, \underbrace{1, \dots, 1}_{|V| - \lfloor \frac{2}{3}|V| \rfloor}, \underbrace{0, \dots, 0}_{\lfloor \frac{1}{3}|V| \rfloor})$.

In all tables, results correspond to a groups of 5 instances (inst$\in \{1, \dots, 5\}$) with the same size $|V|$. Instances with the same pair $(|V|, \text{inst})$ share the same cost structure, independent of other parameter values. We present results for each instance and average results for each group. This way, in total, we have a set of 360 benchmark

Table 3 Theoretical model dimensions

	#*c*	#*v*	#*b*
F^{4i}	$\frac{\lvert V\rvert^4}{4} + \frac{\lvert V\rvert^2}{2} + \frac{5\lvert V\rvert}{2} + 1$	$\frac{\lvert V\rvert^4}{2} + \lvert V\rvert^2 - \lvert V\rvert$	$\lvert V\rvert^3 + \frac{\lvert V\rvert^2}{2} - \frac{\lvert V\rvert}{2}$
F^{3i}	$\frac{\lvert V\rvert^3}{2} + \frac{3\lvert V\rvert^2}{2} + \lvert V\rvert + 1$	$2\lvert V\rvert^3 - \frac{\lvert V\rvert^2}{2} - \frac{\lvert V\rvert}{2}$	$\lvert V\rvert^3 - \frac{\lvert V\rvert^2}{2} - \frac{\lvert V\rvert}{2}$
F^{4ic}	$\frac{\lvert V\rvert^4}{4} + \frac{\lvert V\rvert^2}{4} + \frac{\lvert V\rvert}{2} + \lvert V\rvert\lvert H\rvert + 2$	$\frac{\lvert V\rvert^4}{2} - \lvert V\rvert^3 + \lvert V\rvert^2 - \frac{\lvert V\rvert}{2} + \lvert V\rvert\lvert H\rvert$	$\frac{3\lvert V\rvert^2}{2} - \frac{\lvert V\rvert}{2} + \lvert V\rvert\lvert H\rvert$
F^{3ic}	$\frac{\lvert V\rvert^3}{2} + \frac{3\lvert V\rvert^2}{2} - \lvert V\rvert + \lvert V\rvert\lvert H\rvert + 2$	$\lvert V\rvert^3 + \frac{\lvert V\rvert^2}{2} - \frac{\lvert V\rvert}{2} + \lvert V\rvert\lvert H\rvert$	$\frac{3\lvert V\rvert^2}{2} - \frac{\lvert V\rvert}{2} + \lvert V\rvert\lvert H\rvert$

Table 4 Numerical model dimensions

	$\|V\| = 20$			$\|V\| = 30$			$\|V\| = 40$			$\|V\| = 50$		
	#c	#v	#b	#c	#v	#b	#c	#v	#b	#c	#v	#b
F^{4i}	40,151	80,390	8190	202,801	405,885	27,435	640,501	1,281,580	64,780	1,563,251	3,127,475	126,225
F^{3i}	4621	15,790	8190	14,881	53,535	27,435	34,441	127,180	64,780	66,301	248,725	126,225
F^{4ic}	44,112	76,790	4590	208,742	385,785	7335	648,422	1,227,180	10,380	1,573,152	3,014,975	13,725
F^{3ic}	8582	12,190	4590	20,822	33,435	7335	42,362	72,780	10,380	76,202	136,225	13,725

instances, one per each choice of (|V|, p, α, inst, λ). All instances were solved with the MIP Xpress 8.0 optimizer, under a Windows 7 environment in an Intel(R) Core(TM)i7 CPU 2.93 GHz processor and 16 GB RAM. Default values were initially used for all parameters of Xpress solver and a CPU time limit of 3600 s was set. We have also tested different combinations of parameters for the solver cut strategy and intensity of heuristics but, unless it is specified, the best results were obtained with the parameters of the solver set to the default values.

Tables are grouped in blocks. The first block contains 3 columns with the values of the instances parameters. Then, we give a block of 6 columns for each tested formulation. The columns of each block are the following:

1. Columns |#| indicate the number of instances in the group that could be solved to optimality within the CPU time limit.
2. Columns $g\overline{U}R$ give the percentage relative root gap, computed as $100\frac{\text{obj}_{\overline{U}}-\text{obj}_R}{\text{obj}_{\overline{U}}}$, where $\text{obj}_{\overline{U}}$ denotes the best known upper bound obtained in all our experiments and obj_R denotes the optimal value of the linear relaxation at the root node.
3. Columns $g\overline{U}L$ give the percentage relative lower bound gap, computed as $100\frac{\text{obj}_{\overline{U}}-\text{obj}_L}{\text{obj}_{\overline{U}}}$, obj_L denotes the lower bound at termination.
4. Columns $gU\overline{L}$ give the percentage relative upper bound gap, computed as $100\frac{\text{obj}_U-\text{obj}_{\overline{L}}}{\text{obj}_U}$, where obj_U denotes the upper bound at termination and $\text{obj}_{\overline{L}}$ denotes the best known lower bound obtained in all our experiments.
5. Columns gUL give the percentage relative gap at termination, computed as $100\frac{\text{obj}_U-\text{obj}_L}{\text{obj}_L}$.
6. Columns *nod* indicate the average number of nodes explored in the Branch-and-Bound (B&B) tree search.

Note that, while $g\overline{U}R$ and $g\overline{U}L$ provide quality measures of the lower bounds (at the root node and at termination, respectively), $gU\overline{L}$ provides a quality measure of the upper bounds. In addition, gUL provides a measure of both upper and lower bounds for average performance. Entries with the symbol "–" indicates that the average gaps are 0, or in other words, that all instances were solved to optimality. In addition, entries with the symbol "*" indicates that the average gap could not be computed because the obj_R, obj_L or obj_U was missing in at least one instance of the group. This is usually returned when the solver is not able to load the instance into memory. In addition, obj_R and obj_L are not returned when the solver is not able to solve the linear relaxation of the problem and obj_U is not returned when the solver is not able to load an initial solution and neither able to obtain a feasible solution for the problem. In the computational experiments reported in all our tables, we have used as initial solution the one returned by the solver for the OMT model (see Sect. 2.2) after 5 min of CPU time. Once tree variables (y) and allocation variables (z) are obtained, the sorting and flow variables can be computed.

The caption just above each block gives the formulation the block refers to. We recall that F^{3i}, F^{3ic}, F^{4i}, F^{4ic} stand for the OMTHL formulations 3 index, covering 3 index, 4 index, and covering 4 index, introduced in Sects. 3.2, 3.3, 3.1 and also 3.3, respectively. In addition F^{3ic}&fix, F^{4ic}&fix, stand for the covering 3 index and covering 4 index OMTHL formulations with variable fixing and F^{3ic}&fix&(3), F^{4ic}&fix&(3) also include the valid inequality (3). The reader may note that each table contains 8 blocks (divided into two parts). To facilitate the comparison among tables, the best results in each table are marked in bold so as to highlight the best values among the eight proposed formulations.

Table 5 presents the results for the median criterion. At a first glance, we observe that blocks F^{3i}, F^{3ic}, F^{3ic}&fix and F^{3ic}&fix&(3) improve one by one in sequence. The main difference between blocks F^{3i} and those with covering variables (F^{3ic}, F^{3ic}&fix and F^{3ic}&fix&(3)) is that the latter remain at the root node of the B&B search, when solving instances above 40 nodes, just adding some pre-processing cuts and loading the initial solution. This happens because solving a single node is very costly as it is shown in column *nod*. Even with this limitation, the covering formulations outperforms F^{3i}. In fact, F^{3i} is almost not able to improve the relative root gap. Along the 3-index blocks, we also observe that $g\overline{UL}$ is bigger than $g\underline{UL}$, indicating that the lower bound is in general closer to the optimum than the upper bound. This observation is also supported by the gUL values. Analogously blocks F^{4i}, F^{4ic}, F^{4ic}&fix and F^{4ic}&fix&(3) improve one by one in sequence. Two main differences can be noted between the 4-index and the 3-index formulations. First, the linear relaxation in the 4-index blocks are significantly better (and quite close to the optimum) than those in the 3-index blocks. However, we secondly observe that the 4-index formulations blow up above 40 nodes where only F^{4ic}&fix&(3) block is able to solve the linear relaxation and load an initial solution for 50 nodes instances after one hour of CPU time. In addition, 3-index formulations are able to provide both upper and lower bounds in all blocks but the remaining gaps at termination are quite large (as it is shown in columns $g\overline{UL}$ and $g\underline{UL}$). From this table, we conclude that F^{4ic}&fix&(3) is the best formulation in mostly all cases. In this block, instances solved to optimality require only few nodes (in the B&B search) since the linear relaxation provides a lower bound very close to the optimum.

Table 6 presents the results for the k-centrum criterion. As in the median criterion, the reader may observe that covering formulations fixing variables provide the best performance. However, it can be noticed that k-centrum criterion is much more difficult to solve than median criterion, as it can be observed in terms of gaps and instances solved to optimality. In particular, no instance above 20 nodes was solved to optimality. In this case, the pre-processing for fixing variables and the family of valid inequalities (3) allow to solve the linear relaxation of a larger number of instances for F^{4ic}&fix&(3). However, F^{4ic}&fix has a better performance than F^{4ic}&fix&(3) in mostly all cases if the linear relaxation can be solved. From this table, we conclude that covering formulations fixing variables are the best formulations for almost all instances but still the linear relaxation of some 50 nodes instances cannot be solved after 1 hour of CPU time.

Table 7 presents the results for the k-trimmed mean criterion. Similar observations to those for the median and k-centrum criteria can be made in this case. As in the previous criteria, F^{4ic}&fix&(3) provides the best performance in terms of gaps (intances of 20–40 nodes) and instances solved to optimality ($|V| = 20$). However,

Table 5 OMTHL results for the median criterion

$\|V\|$	α	p	F^{3i}						F^{3ic}						F^{3ic} &fix						F^{3ic} &fix&(3)					
			|#|	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	|#|	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	|#|	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	|#|	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod
20	0.5	3	**5**	12.6	–	–	–	**2e3**	**5**	12.6	–	–	–	1e2	**5**	12.6	–	–	–	1e2	**5**	12.6	–	–	–	1e2
20	0.5	5	0	24.9	19.9	11.7	29.3	2e3	4	24.9	4.0	0.4	4.2	**6e3**	4	24.9	4.1	1.0	4.9	4e3	4	24.9	3.7	0.9	4.4	4e3
20	0.5	8	0	37.8	33.7	16.2	44.5	3e3	0	37.8	25.0	12.1	34.2	8e3	0	37.8	25.9	10.9	34.0	6e3	0	37.8	25.2	11.9	34.1	**9e3**
20	0.8	3	**5**	16.2	–	–	–	**4e3**	**5**	16.2	–	–	–	8e2	**5**	16.2	–	–	–	6e2	**5**	16.2	–	–	–	7e2
20	0.8	5	0	30.6	25.9	9.4	32.9	4e3	4	30.6	5.3	1.1	6.1	**9e3**	2	30.6	9.3	2.3	11.1	7e3	4	30.6	4.8	1.6	6.0	9e3
20	0.8	8	0	44.9	40.8	16.4	50.5	3e3	0	44.9	32.4	12.0	40.5	9e3	0	44.9	32.1	11.2	39.7	7e3	0	44.9	31.6	13.0	40.4	**1e4**
30	0.5	3	0	11.6	9.6	8.9	17.2	**3e4**	**3**	11.6	2.3	**0.6**	2.3	3e2	**3**	11.5	2.6	0.7	2.7	4e2	**3**	11.5	2.8	1.0	3.2	2e2
30	0.5	5	0	20.5	18.5	15.7	31.4	1e2	0	20.5	12.8	5.4	17.5	7e2	0	20.5	13.4	6.0	18.5	8e2	0	20.5	12.7	3.9	16.1	**1e3**
30	0.5	8	0	30.6	29.3	16.6	40.8	1e2	0	30.6	23.3	15.0	34.6	7e2	0	30.6	23.2	11.8	32.0	6e2	0	30.6	23.2	15.0	34.5	**9e2**
30	0.8	3	0	12.8	10.5	16.2	24.7	**2e4**	3	12.8	2.8	1.0	3.4	2e2	**4**	12.7	2.2	0.9	2.6	2e2	**4**	12.7	2.1	1.2	2.8	8e2
30	0.8	5	0	24.3	22.5	26.1	42.9	2e2	0	24.3	16.4	7.0	22.1	7e2	0	24.3	16.3	7.3	22.3	8e2	0	24.3	15.8	8.3	22.7	**8e2**
30	0.8	8	0	36.6	35.2	27.1	52.4	2e2	0	36.6	28.9	20.0	42.7	1e3	0	36.6	28.8	19.7	42.4	7e2	0	36.6	28.9	23.4	45.1	**1e3**
40	0.5	3	0	12.2	11.5	17.6	25.2	**1e3**	0	12.2	7.1	7.8	12.0	0	0	12.1	7.2	16.7	20.7	0	0	12.1	7.3	17.9	22.0	0
40	0.5	5	0	22.7	22.2	19.7	34.5	**12**	0	22.7	17.2	16.3	27.3	0	0	22.7	17.2	18.0	28.9	0	0	22.7	17.3	20.3	30.9	0
40	0.5	8	**0**	33.2	32.6	25.2	46.1	**13**	**0**	33.2	26.9	21.4	38.6	1	**0**	33.2	27.0	22.6	39.5	0	**0**	33.2	27.0	29.0	44.6	0
40	0.8	3	0	9.9	9.2	28.2	34.8	**1e3**	0	9.9	3.8	7.2	10.6	0	0	9.9	4.0	25.5	28.5	0	0	9.9	4.1	28.4	31.4	0
40	0.8	5	0	22.1	21.3	30.6	45.1	**9**	0	22.1	15.4	24.7	36.0	0	0	22.1	15.5	26.2	37.3	0	0	22.1	15.7	30.6	41.2	0
40	0.8	8	0	33.9	33.2	34.0	55.6	**12**	0	33.9	27.0	28.7	47.6	4	0	33.9	27.1	29.4	48.2	1	0	33.9	27.1	38.4	54.8	0
50	0.5	3	**0**	22.5	22.4	13.2	23.0	**1**	**0**	22.5	19.2	**12.6**	19.2	0	**0**	22.5	19.1	**12.6**	19.1	0	**0**	22.5	19.5	13.4	20.3	0
50	0.5	5	0	33.1	33.1	26.0	39.3	**1**	0	33.1	29.6	23.8	34.3	0	0	33.1	29.5	23.8	34.2	0	0	33.1	29.8	27.1	37.3	0
50	0.5	8	**0**	47.3	47.2	**28.6**	47.7	**3**	**0**	47.3	43.8	28.8	44.4	0	**0**	47.3	43.7	28.8	44.4	0	**0**	47.3	43.9	32.0	47.1	0
50	0.8	3	0	22.9	22.9	21.7	31.7	**0**	0	22.9	19.1	18.7	25.6	**0**	0	22.9	19.0	17.9	24.7	**0**	0	22.9	19.5	22.0	28.9	**0**
50	0.8	5	0	39.7	39.6	36.1	50.2	**1**	0	39.7	35.8	29.8	41.7	0	0	39.7	35.7	29.8	41.6	0	0	39.7	35.9	37.3	48.1	0
50	0.8	8	**0**	55.8	55.7	42.4	59.1	**3**	**0**	55.8	52.2	**37.6**	52.2	0	**0**	55.8	52.2	**37.6**	52.2	0	**0**	55.8	52.3	45.0	57.9	0

Table 5 (continued)

$\|V\|$	α	p	F^{4i}						F^{4ic}						F^{4ic}&*fix*						F^{4ic}&*fix*&(3)					
			$\|\#\|$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	*nod*	$\|\#\|$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	*nod*	$\|\#\|$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	*nod*	$\|\#\|$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	*nod*
20	0.5	3	**5**	–	–	–	–	5e2	**5**	–	–	–	–	1	**5**	–	–	–	–	1	**5**	–	–	–	–	1
20	0.5	5	**5**	**0.4**	–	–	–	1e2	**5**	**0.4**	–	–	–	3	**5**	**0.4**	–	–	–	3	**5**	**0.4**	–	–	–	2
20	0.5	8	**5**	**0.2**	–	–	–	17	**5**	**0.2**	–	–	–	3	**5**	**0.2**	–	–	–	3	**5**	**0.2**	–	–	–	2
20	0.8	3	**5**	–	–	–	–	1e3	**5**	–	–	–	–	0	**5**	–	–	–	–	0	**5**	–	–	–	–	0
20	0.8	5	**5**	**0.4**	–	–	–	8e2	**5**	**0.4**	–	–	–	3	**5**	**0.4**	–	–	–	13	**5**	**0.4**	–	–	–	2
20	0.8	8	**5**	**0.9**	–	–	–	2e2	**5**	**0.9**	–	–	–	30	**5**	**0.9**	–	–	–	18	**5**	**0.9**	–	–	–	20
30	0.5	3	0	**0.8**	0.7	14.5	14.6	3	2	**0.8**	0.7	1.2	**1.3**	0	2	**0.8**	0.7	1.5	1.6	0	**3**	**0.8**	**0.6**	4.1	4.2	15
30	0.5	5	0	**1.6**	1.0	17.2	18.0	17	4	**1.6**	0.2	1.7	1.9	2e2	3	**1.6**	0.2	0.1	0.4	87	**5**	**1.6**	–	–	–	36
30	0.5	8	1	**2.2**	1.5	10.6	11.7	9	2	**2.2**	0.8	0.6	1.0	74	3	**2.2**	0.7	**0.4**	0.7	50	**4**	**2.2**	**0.5**	**0.4**	**0.5**	83
30	0.8	3	0	**0.5**	0.5	23.7	23.7	24	**4**	**0.5**	**0.4**	**0.4**	**0.4**	0	3	**0.5**	**0.4**	3.0	3.0	53	**4**	**0.5**	**0.4**	0.8	0.8	27
30	0.8	5	1	**0.7**	0.4	21.1	21.4	8	**4**	**0.7**	**0.1**	**0.1**	**0.1**	42	**4**	**0.7**	0.3	1.2	1.4	66	**4**	**0.7**	**0.1**	3.5	3.5	36
30	0.8	8	0	**2.1**	1.1	22.7	23.0	80	**4**	**2.1**	**0.9**	1.0	**1.0**	25	**4**	**2.1**	**0.9**	1.1	1.1	1e2	**4**	**2.1**	1.0	**0.9**	**1.0**	27
40	0.5	3	0	*	*	*	*	0	**4**	*	*	*	*	0	**4**	**2.6**	**2.6**	**4.2**	**4.2**	0	**4**	**2.6**	**2.6**	4.3	4.3	0
40	0.5	5	0	*	*	*	*	0	**3**	*	*	*	*	0	**3**	**4.6**	**4.6**	**4.6**	**4.6**	0	**3**	**4.6**	**4.6**	8.8	8.8	0
40	0.5	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**6.5**	**6.4**	12.7	12.7	0	**0**	**6.5**	**6.4**	**12.2**	**12.3**	0
40	0.8	3	0	*	*	*	*	0	4	*	*	*	*	0	**5**	–	–	–	–	1	**5**	–	–	–	–	1
40	0.8	5	0	*	*	*	*	0	**4**	**0.6**	**0.4**	0.7	0.7	2	**4**	**0.6**	0.6	0.5	**0.6**	0	**4**	**0.6**	0.6	**0.4**	**0.6**	0
40	0.8	8	0	*	*	*	*	0	**4**	**0.6**	**0.6**	1.0	1.0	0	**4**	**0.6**	**0.6**	**0.6**	**0.6**	1	**4**	**0.6**	**0.6**	0.8	0.8	1
50	0.5	3	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**15.4**	**12.6**	13.4	**13.4**	0
50	0.5	5	0	*	*	*	*	0	0	*	*	*	*	0	0	*	*	*	*	0	**1**	**18.3**	**18.3**	**21.2**	**21.2**	0
50	0.5	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**28.0**	**28.0**	31.5	**31.5**	0
50	0.8	3	0	*	*	*	*	**0**	0	*	*	*	*	**0**	0	*	*	*	*	**0**	**2**	**11.6**	**11.6**	**12.8**	**12.8**	**0**
50	0.8	5	0	*	*	*	*	0	0	*	*	*	*	0	0	*	*	*	*	0	**1**	**22.8**	**22.8**	**29.1**	**29.1**	0
50	0.8	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**38.2**	**37.6**	44.1	**44.1**	0

Table 6 OMTHL results for the k-centrum criterion

$\|V\|$	α	p	F^{3i}						F^{3ic}						F^{3ic}&*fix*						F^{3ic}&*fix*&(3)					
			$\|\#\|$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	*nod*	$\|\#\|$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	*nod*	$\|\#\|$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	*nod*	$\|\#\|$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	*nod*
20	0.5	3	0	45.6	40.7	10.3	44.5	**1e4**	1	45.6	17.4	5.9	18.8	7e3	1	26.5	6.4	**4.2**	6.4	7e3	0	26.4	11.3	4.6	11.7	6e3
20	0.5	5	0	56.2	51.2	17.8	59.1	5e3	0	56.2	29.2	7.0	32.9	7e3	0	39.2	24.8	7.1	28.6	**1e4**	0	39.2	26.2	6.3	29.5	7e3
20	0.5	8	0	66.4	63.3	22.6	71.2	5e3	0	66.4	37.6	14.5	45.8	9e3	0	50.6	34.8	15.1	43.7	**1e4**	0	50.6	35.0	14.3	43.3	9e3
20	0.8	3	0	46.5	40.1	9.4	44.6	**1e4**	1	46.5	18.9	3.9	20.2	8e3	1	28.9	9.9	4.0	11.4	7e3	2	28.9	8.1	2.9	8.7	5e3
20	0.8	5	0	57.5	52.0	30.1	65.6	9e3	0	57.5	31.7	8.8	36.1	**1e4**	0	43.0	23.8	11.7	30.7	1e4	0	43.0	28.3	10.9	34.5	7e3
20	0.8	8	0	67.4	64.3	29.3	74.3	7e3	0	67.4	40.8	15.7	49.2	1e4	0	54.9	38.9	13.5	46.1	**1e4**	0	54.9	41.0	21.7	52.8	1e4
30	0.5	3	**0**	43.3	41.5	34.4	56.2	**5e3**	**0**	43.3	28.4	17.0	32.2	4e2	**0**	22.3	14.1	15.3	16.9	9e2	**0**	22.3	15.4	14.3	17.3	3e2
30	0.5	5	**0**	50.8	48.9	41.0	65.5	85	**0**	50.8	35.4	34.6	51.6	8e2	**0**	32.2	24.8	35.0	44.1	**9e2**	**0**	32.2	23.9	30.3	39.3	5e2
30	0.5	8	**0**	60.4	59.2	42.3	72.3	2e2	**0**	60.4	42.6	37.0	57.4	1e3	**0**	43.6	35.8	33.0	49.4	**1e3**	**0**	43.6	36.3	40.2	55.4	9e2
30	0.8	3	**0**	43.4	39.9	37.2	57.4	**5e3**	**0**	43.4	27.0	16.7	31.1	6e2	**0**	23.8	14.5	19.0	21.6	6e2	**0**	23.8	15.1	15.1	18.5	4e2
30	0.8	5	**0**	51.6	48.7	51.5	71.2	1e2	**0**	51.6	35.7	40.0	55.1	5e2	**0**	35.5	27.6	39.2	49.0	**1e3**	**0**	35.5	27.5	52.1	59.8	7e2
30	0.8	8	**0**	60.7	59.3	52.1	77.0	4e2	**0**	60.7	45.7	43.0	63.6	8e2	**0**	46.7	38.8	40.0	56.7	**1e3**	**0**	46.7	38.5	50.6	64.3	1e3
40	0.5	3	**0**	48.4	48.2	41.1	60.6	**2e2**	**0**	48.4	42.8	34.2	51.3	0	**0**	28.7	25.2	**25.0**	**27.1**	0	**0**	28.7	25.3	41.5	43.4	0
40	0.5	5	**0**	61.7	61.4	45.8	66.9	**8**	**0**	61.7	56.2	44.1	61.2	0	**0**	47.1	43.5	41.9	48.0	2	**0**	47.1	43.8	46.4	52.2	0
40	0.5	8	**0**	70.1	69.5	49.2	73.8	11	**0**	70.1	64.0	45.1	66.4	0	**0**	58.2	53.8	45.2	57.1	**13**	**0**	58.2	54.7	50.9	62.3	8
40	0.8	3	**0**	48.0	47.6	51.4	67.2	**4e2**	**0**	48.0	41.6	42.3	56.5	0	**0**	29.4	25.0	**24.0**	**26.1**	2	**0**	29.4	25.3	34.7	36.6	1
40	0.8	5	**0**	64.2	63.7	56.6	73.2	**7**	**0**	64.2	58.6	45.4	61.5	0	**0**	51.9	47.7	**44.9**	51.0	6	**0**	51.9	48.4	51.6	57.4	0
40	0.8	8	**0**	72.6	72.0	61.0	80.0	13	**0**	72.6	66.9	53.4	71.6	2	**0**	63.0	59.3	52.4	64.4	**21**	**0**	63.0	59.3	62.3	71.9	16
50	0.5	3	**0**	55.3	55.3	37.1	58.4	**0**	**0**	55.3	51.8	**32.5**	51.9	**0**	**0**	37.9	35.7	32.6	35.9	**0**	**0**	37.9	35.7	37.2	40.3	**0**
50	0.5	5	**0**	66.8	66.8	50.8	70.6	**0**	**0**	66.8	63.3	46.4	64.6	**0**	**0**	53.7	51.6	**45.1**	52.1	**0**	**0**	53.7	51.7	50.9	57.3	**0**
50	0.5	8	**0**	73.5	73.5	56.2	76.2	**0**	**0**	73.5	69.7	52.9	70.6	**0**	**0**	62.2	60.1	**52.2**	**60.9**	**0**	**0**	62.2	60.2	57.3	65.1	**0**
50	0.8	3	**0**	58.4	58.4	47.1	65.1	**0**	**0**	58.4	54.6	38.3	55.8	**0**	**0**	43.4	40.8	38.7	42.6	**0**	**0**	43.3	41.0	47.1	50.5	**0**
50	0.8	5	**0**	69.9	69.9	60.9	76.9	**0**	**0**	69.9	66.4	50.1	67.1	**0**	**0**	59.0	56.9	**49.5**	57.2	**0**	**0**	59.0	56.9	61.2	67.1	**0**
50	0.8	8	**0**	77.3	77.3	65.3	81.6	**0**	**0**	77.3	73.8	58.3	74.5	**0**	**0**	68.6	66.6	**57.9**	**67.0**	**0**	**0**	68.6	66.6	66.3	73.7	**0**

Table 6 (continued)

$\lvert V\rvert$	α	p	F^{4i}						F^{4ic}						F^{4ic}&*fix*						F^{4ic}&*fix*&(3)					
			$\lvert\#\rvert$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	$\lvert\#\rvert$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	$\lvert\#\rvert$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	$\lvert\#\rvert$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod
20	0.5	3	0	33.7	28.1	10.6	33.0	1e3	1	33.7	14.5	6.2	16.4	7e2	2	**17.5**	6.3	9.2	11.1	1e3	**3**	**17.5**	**4.2**	4.9	**4.9**	5e2
20	0.5	5	0	31.7	27.5	10.6	34.0	8e2	0	31.7	10.5	3.6	12.2	1e3	**4**	**18.3**	**1.9**	2.2	2.2	9e2	**4**	**18.3**	**1.9**	**1.9**	**1.9**	1e3
20	0.5	8	0	31.9	28.7	14.6	38.1	6e2	3	31.9	5.5	**1.8**	5.6	1e3	**4**	**18.7**	**1.9**	**1.8**	**2.0**	1e3	3	**18.7**	2.9	1.9	3.1	1e3
20	0.8	3	0	33.1	27.8	6.4	30.9	1e3	0	33.1	13.9	3.9	15.4	9e2	**3**	**19.0**	**3.6**	2.7	**4.1**	1e3	2	**19.0**	4.7	**2.4**	4.9	1e3
20	0.8	5	0	31.0	26.4	19.8	39.8	7e2	2	31.0	7.1	3.1	7.7	1e3	**4**	**20.6**	**2.5**	2.8	**2.9**	1e3	3	**20.6**	3.5	**2.7**	3.7	1e3
20	0.8	8	0	31.7	27.7	11.7	35.2	6e2	3	31.7	3.7	2.6	4.3	1e3	**4**	**21.3**	**2.0**	**2.0**	**2.0**	1e3	**4**	**21.3**	2.1	2.5	2.6	1e3
30	0.5	3	**0**	34.1	33.6	32.2	48.7	0	**0**	34.1	28.7	25.0	38.9	10	**0**	**16.3**	**13.0**	**12.5**	**13.2**	9	**0**	**16.3**	13.6	15.7	16.9	5
30	0.5	5	**0**	*	*	*	*	0	**0**	31.2	22.8	16.4	26.1	10	**0**	**17.5**	**12.9**	**12.8**	**13.0**	20	**0**	**17.5**	13.5	13.2	14.0	29
30	0.5	8	**0**	*	*	*	*	0	**0**	32.8	22.5	16.6	24.0	34	**0**	**20.6**	**15.6**	**16.2**	**16.9**	45	**0**	**20.6**	15.7	17.2	18.0	29
30	0.8	3	**0**	33.2	31.9	48.9	60.5	0	**0**	33.2	27.6	17.1	32.3	5	**0**	**16.6**	**12.1**	**11.8**	**12.3**	34	**0**	**16.6**	13.2	**11.8**	13.4	11
30	0.8	5	**0**	30.3	30.0	45.5	55.5	0	**0**	30.3	22.2	14.6	22.7	6	**0**	**19.5**	**14.5**	15.1	**15.5**	24	**0**	**19.5**	15.5	**14.5**	16.0	28
30	0.8	8	**0**	31.4	31.0	44.3	54.7	0	**0**	31.4	18.2	20.1	22.7	23	**0**	**21.6**	**15.8**	**15.9**	**16.2**	56	**0**	**21.6**	16.1	17.5	18.1	31
40	0.5	3	**0**	*	*	*	*	0	**0**	42.7	41.0	39.1	53.5	0	**0**	**24.1**	**23.3**	38.1	38.3	0	**0**	**24.1**	**23.3**	41.5	41.7	0
40	0.5	5	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**38.2**	**36.8**	**38.0**	**38.0**	0	**0**	**38.2**	**36.8**	40.6	40.6	0
40	0.5	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**42.5**	41.1	**41.9**	**42.1**	1	**0**	**42.5**	**40.9**	46.8	46.9	0
40	0.8	3	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**24.1**	**23.1**	51.9	52.1	0
40	0.8	5	**0**	*	*	*	*	0	**0**	51.9	49.7	49.1	56.5	0	**0**	**42.7**	**41.3**	49.6	**49.6**	0	**0**	**42.7**	**41.3**	57.2	57.2	0
40	0.8	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**47.6**	46.1	**48.6**	**48.6**	0	**0**	**47.6**	**46.0**	63.3	63.3	0
50	0.5	3	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	**32.4**	**32.4**	32.8	**32.8**	**0**	**0**	**32.4**	**32.4**	35.9	35.9	**0**
50	0.5	5	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	**44.6**	**44.5**	45.5	**45.5**	**0**	**0**	**44.6**	44.6	47.2	47.2	**0**
50	0.5	8	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**
50	0.8	3	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	**36.7**	**36.7**	**38.2**	**38.2**	**0**	**0**	**36.7**	**36.7**	47.1	47.1	**0**
50	0.8	5	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	**49.1**	**49.1**	57.1	**57.1**	**0**
50	0.8	8	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**

Table 7 OMTHL results for the k-trimmed mean criterion

$\lvert V\rvert$	α	p	F^{3i}						F^{3ic}						F^{3ic}&fix						F^{3ic}&fix&(3)					
			$\lvert\#\rvert$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	$\lvert\#\rvert$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	$\lvert\#\rvert$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	$\lvert\#\rvert$	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod
20	0.5	3	0	45.1	37.9	6.8	42.1	**1e4**	4	45.1	3.9	-	3.9	8e3	4	27.9	3.4	0.1	3.5	5e3	4	27.8	2.7	0.1	2.9	6e3
20	0.5	5	0	57.6	52.1	22.2	62.8	6e3	0	57.6	18.7	6.1	23.4	**2e4**	0	41.7	23.5	7.2	28.9	1e4	0	41.7	24.3	4.9	28.1	1e4
20	0.5	8	0	67.2	61.1	18.2	68.1	6e3	0	67.2	40.6	11.3	47.2	1e4	0	57.7	38.9	17.1	49.0	2e4	0	57.7	38.5	17.9	49.2	**2e4**
20	0.8	3	0	46.8	37.9	5.6	41.1	**1e4**	4	46.8	4.4	0.2	4.6	9e3	3	31.2	7.9	0.5	8.3	6e3	4	31.2	4.6	-	4.6	7e3
20	0.8	5	0	61.7	55.4	23.4	65.9	8e3	0	61.7	26.9	6.8	31.7	**2e4**	0	49.2	30.7	4.5	33.7	1e4	0	49.2	29.8	8.1	35.3	1e4
20	0.8	8	0	70.4	64.2	14.2	69.3	7e3	0	70.4	46.3	17.8	55.8	2e4	0	63.6	44.0	19.3	54.6	**3e4**	0	63.6	46.6	17.4	55.6	2e4
30	0.5	3	**0**	41.6	38.5	44.7	62.6	**6e3**	**0**	41.6	24.0	19.0	31.9	1e3	**0**	24.3	14.9	20.6	25.4	1e3	**0**	24.2	15.4	27.6	32.3	9e2
30	0.5	5	**0**	56.1	53.9	41.5	69.5	5e2	**0**	56.1	36.0	23.2	44.2	**2e3**	**0**	41.0	30.4	24.7	40.3	1e3	**0**	41.0	30.5	33.0	47.3	9e2
30	0.5	8	0	65.6	63.5	34.9	74.7	4e2	0	65.6	41.2	23.8	52.4	**2e3**	0	51.5	41.4	34.3	59.2	1e3	0	51.5	40.1	32.9	57.3	1e3
30	0.8	3	**0**	43.0	39.1	46.6	63.3	**2e3**	**0**	43.0	25.9	13.1	27.8	1e3	**0**	27.6	18.1	15.3	22.3	8e2	**0**	27.5	17.5	16.1	22.5	8e2
30	0.8	5	**0**	56.5	54.3	54.3	76.7	8e2	**0**	56.5	35.5	21.5	43.5	**3e3**	**0**	43.4	32.4	39.9	54.9	1e3	**0**	43.4	32.0	33.5	49.7	1e3
30	0.8	8	**0**	68.0	65.9	39.4	77.7	5e2	**0**	68.0	48.1	30.0	60.6	**3e3**	**0**	57.2	46.4	40.2	65.5	1e3	**0**	57.2	46.0	41.9	66.1	1e3
40	0.5	3	**0**	38.6	37.8	49.3	64.0	**1e3**	**0**	38.6	32.5	37.1	51.6	1	**0**	20.1	15.5	40.9	43.0	2	**0**	20.1	15.3	38.5	40.4	0
40	0.5	5	**0**	50.1	49.4	55.4	73.6	**11**	**0**	50.1	40.5	39.8	58.2	2	**0**	33.8	26.9	55.7	62.2	5	**0**	33.8	27.1	48.9	56.4	5
40	0.5	8	**0**	64.2	63.1	57.5	80.6	12	**0**	64.2	50.3	43.8	65.4	5	**0**	50.1	42.4	51.2	65.3	**22**	**0**	50.1	42.8	57.7	70.1	14
40	0.8	3	**0**	38.3	36.7	61.1	71.9	**6e2**	**0**	38.3	31.3	32.9	47.3	1	**0**	21.2	15.2	24.2	26.6	9	**0**	21.2	15.5	37.9	40.3	1
40	0.8	5	**0**	51.4	49.9	65.1	79.3	**11**	**0**	51.4	40.6	46.0	62.1	5	**0**	37.1	29.2	52.4	60.2	10	**0**	37.1	29.0	60.2	66.9	9
40	0.8	8	**0**	64.5	63.7	66.0	85.1	17	**0**	64.5	50.9	50.2	70.6	16	**0**	52.5	45.0	61.7	74.4	24	**0**	52.5	45.2	58.8	72.7	**26**
50	0.5	3	**0**	53.8	53.7	44.4	61.4	**2**	**0**	53.8	50.0	**33.3**	50.0	0	**0**	40.2	37.4	44.8	48.1	0	**0**	40.1	37.2	44.8	48.0	0
50	0.5	5	**0**	68.7	68.7	58.1	75.5	**0**	**0**	68.7	65.1	**46.5**	65.1	**0**	**0**	58.7	55.8	59.7	66.7	**0**	**0**	58.7	55.9	59.7	66.7	**0**
50	0.5	8	**0**	78.1	78.0	68.8	81.6	**1**	**0**	78.1	74.7	**63.2**	**75.3**	0	**0**	**70.4**	**67.9**	71.1	**75.3**	0	**0**	**70.4**	**67.9**	71.1	**75.3**	0
50	0.8	3	**0**	57.5	57.3	54.7	68.9	**9**	**0**	57.5	53.4	**37.7**	**53.4**	0	**0**	45.8	42.8	55.5	59.0	0	**0**	45.7	42.5	55.5	58.9	0
50	0.8	5	**0**	72.9	72.9	67.7	81.7	**0**	**0**	72.9	69.3	**52.3**	69.3	**0**	**0**	65.0	62.1	68.9	75.3	**0**	**0**	65.0	62.1	68.9	75.3	**0**
50	0.8	8	**0**	81.7	81.6	77.9	87.3	**1**	**0**	81.7	78.5	**67.5**	**78.5**	0	**0**	**75.9**	**73.6**	78.4	82.2	0	**0**	**75.9**	**73.6**	78.4	82.2	0

Table 7 (continued)

$\|V\|$	α	p	F^{4i}						F^{4ic}						F^{4ic} &fix						F^{4ic} &fix&(3)					
			|#|	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	|#|	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	|#|	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod	|#|	$g\overline{U}R$	$g\overline{U}L$	$gU\overline{L}$	gUL	nod
20	0.5	3	0	26.9	20.2	2.0	21.8	9e2	3	26.9	2.0	0.4	2.4	1e3	**5**	12.7	–	–	–	5e2	**5**	**12.6**	–	–	–	6e2
20	0.5	5	0	24.4	17.0	4.0	20.2	5e2	**5**	24.4	–	–	–	5e2	**5**	**9.1**	–	–	–	5e2	**5**	**9.1**	–	–	–	4e2
20	0.5	8	0	15.5	7.2	2.1	9.0	6e2	**5**	15.5	–	–	–	20	**5**	**6.3**	–	–	–	5	**5**	**6.3**	–	–	–	12
20	0.8	3	0	26.8	16.4	2.7	18.6	8e2	3	26.8	3.8	0.4	4.1	6e2	3	15.0	2.3	–	2.3	6e2	**5**	**14.9**	–	–	–	4e2
20	0.8	5	0	27.0	19.3	7.7	25.5	7e2	**5**	27.0	–	–	–	6e2	4	**16.7**	0.4	0.1	0.4	5e2	4	**16.7**	0.3	0.8	1.1	1e3
20	0.8	8	0	19.0	11.1	6.8	17.0	9e2	**5**	19.0	–	–	–	3e2	**5**	**12.9**	–	–	–	3e2	**5**	**12.9**	–	–	–	3e2
30	0.5	3	**0**	27.5	26.1	45.1	55.2	0	**0**	27.5	20.3	10.8	21.7	22	**0**	**13.9**	11.0	**9.6**	11.5	16	**0**	**13.9**	**9.3**	10.7	**10.8**	30
30	0.5	5	**0**	30.8	30.0	44.4	55.9	0	**0**	30.8	18.7	**12.9**	19.7	49	**0**	**18.4**	13.1	13.1	14.2	51	**0**	**18.4**	**12.5**	13.1	**13.7**	52
30	0.5	8	0	27.5	26.4	39.2	52.6	0	0	27.5	12.7	6.9	13.7	90	0	**14.8**	9.0	**6.0**	9.1	26	**1**	**14.8**	**6.2**	6.5	**6.9**	1e2
30	0.8	3	**0**	28.2	26.5	49.6	58.3	0	**0**	28.2	19.1	**12.4**	20.6	59	**0**	**16.8**	12.1	20.1	21.5	36	**0**	**16.8**	**11.3**	13.4	**14.0**	36
30	0.8	5	**0**	29.0	27.6	61.1	68.3	10	**0**	29.0	16.7	11.4	17.5	72	**0**	**19.3**	13.9	11.5	14.8	73	**0**	**19.3**	**10.5**	**11.3**	**11.3**	1e2
30	0.8	8	**0**	*	*	*	*	0	**0**	30.0	9.9	8.6	10.9	1e2	**0**	**21.3**	8.9	8.9	10.2	1e2	**0**	**21.3**	**8.5**	**8.3**	**9.1**	1e2
40	0.5	3	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**12.9**	12.5	20.5	20.5	2	**0**	**12.9**	**12.4**	**12.9**	**12.9**	5
40	0.5	5	**0**	*	*	*	*	0	**0**	*	*	*	*	3	**0**	**15.1**	14.4	**14.3**	**14.4**	3	**0**	**15.1**	**14.3**	14.6	14.6	7
40	0.5	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**20.6**	19.8	**23.5**	**23.9**	2	**0**	**20.6**	**19.7**	24.1	24.5	4
40	0.8	3	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**13.0**	12.5	**12.4**	12.5	4	**0**	**13.0**	**12.4**	**12.4**	**12.4**	4
40	0.8	5	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**16.4**	15.9	**15.4**	16.0	5	**0**	16.6	**15.5**	**15.4**	**15.5**	7
40	0.8	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	19.3	17.6	**16.9**	17.9	2	**0**	**19.2**	**16.7**	17.0	**17.1**	6
50	0.5	3	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	**33.3**	**33.3**	44.8	**44.8**	0
50	0.5	5	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	**46.6**	**46.5**	59.7	**59.7**	**0**
50	0.5	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0
50	0.8	3	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	37.9	37.8	55.5	55.6	0	**0**	**37.8**	**37.7**	55.5	55.5	0
50	0.8	5	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	*	*	*	*	**0**	**0**	**52.4**	**52.3**	68.9	**68.9**	**0**
50	0.8	8	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0	**0**	*	*	*	*	0

Table 8 Best OMTHL model for each combination of |V|, α, p and criterion according to $g\overline{UL}$ values

$\|V\|$	α	p	Median	k-centrum	k-trimmed
20	0.5	3	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)
20	0.5	5	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)
20	0.5	8	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
20	0.8	3	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
20	0.8	5	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}
20	0.8	8	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
30	0.5	3	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
30	0.5	5	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
30	0.5	8	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
30	0.8	3	F^{4ic}	F^{4ic}&fix	F^{4ic}&fix&(3)
30	0.8	5	F^{4ic}	F^{4ic}&fix	F^{4ic}&fix&(3)
30	0.8	8	F^{4ic}	F^{4ic}&fix	F^{4ic}&fix&(3)
40	0.5	3	F^{4ic}&fix	F^{4ic}&fix	F^{4ic}&fix&(3)
40	0.5	5	F^{4ic}&fix	F^{4ic}&fix	F^{4ic}&fix&(3)
40	0.5	8	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)
40	0.8	3	F^{4ic}&fix	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)
40	0.8	5	F^{4ic}	F^{4ic}&fix	F^{4ic}&fix&(3)
40	0.8	8	F^{4ic}&fix	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)
50	0.5	3	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
50	0.5	5	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
50	0.5	8	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)	F^{3ic}&fix&(3)
50	0.8	3	F^{4ic}&fix&(3)	F^{4ic}&fix	F^{4ic}&fix&(3)
50	0.8	5	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)
50	0.8	8	F^{4ic}&fix&(3)	F^{4ic}&fix&(3)	F^{3ic}&fix&(3)

when the size of the 4-index formulation is too big to be loaded into memory ($|V| = 50$), only the 3-index formulations are able to solve the linear relaxation and to load the initial solution but leaving significant termination gaps in this case.

To help understanding the conclusions, Table 8 identifies which models work better for each combination of |V|, α, p and criterion according to $g\overline{UL}$ values. We use $g\overline{UL}$ to break ties as a first criterion and the average $g\overline{UL}$ for the |V| group as a second criterion (average $g\overline{UL}$ for all rows, otherwise). From this table, one can conclude that F^{4ic}&fix&(3) is the formulation with the best performance in most of the analyzed instances.

6 Conclusions and future remarks

In this paper, we consider the OMTHL, that is a single-allocation hub location problem where p hubs must be placed on a network and connected by a non-directed tree. The OMTHL is a complex network design problem that involves a number of components (hub network connectivity, flow between each vertex pair, and sorting of the distribution and collection costs) each of which is by itself a hard combinatorial optimization problem. The in/exclusion of these components give rise to different subproblems of the OMTHL that are well-known problems of the literature. We have presented a general 4-index OMTHL formulation that includes all these subproblems. This formulation admits a 3-index reformulation that uses 3-indexes flow variables instead of 4 indexes at the expense of a weaker linear relaxation. Given that ordered median hub location problems with the given sorting variables are rather difficult to solve we have improved the formulations by introducing covering variables in two valid OMTHL reformulations. In addition, we have developed two pre-processings to reduce the size of these formulations. Computational results are given for three criteria (median, k-centrum, k-trimmed mean) showing that F^{4ic}&fix&(3) is the best formulation able to solve instances to optimality (or close to optimality) for sizes up to 30 nodes. F^{4ic}&fix&(3) also provides the best results for instances of 40 nodes but in this case only the linear relaxation can be computed with some additional pre-processing cuts loading an initial solution obtained from the OMT problem. For the largest instances of 50 nodes the best performance is obtained by F^{3ic}&fix&(3). The reader can observe that in these cases after adding cuts and the preprocessing phase, we can solve the linear relaxation of all instances and provide upper bounds based on the initial solutions provided by some of our subproblems.

Acknowledgements The authors have been partially supported by projects MTM2016-74983-C02-01,02 (MINECO/FEDER), CEI-3-FQM331, US-1256951, "NetmeetData" (Fundación BBVA, 2019), P18-FR-1422 (Proyecto Frontera del conocimiento JJAA) and FEDER-UCA18-106895 (Junta de Andalucía/FEDER/UCA).

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